Segment position checks for segment/segment CCD

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1 Motivation

In order for two line segments (with or without thickness) to be colliding, it is necessary for the straight lines to be colliding, and **also** for the segments to be positioned near the intersection of the lines (rather than somewhere far out). Considering the line/line condition is not enough, since it can produce vast amounts of false positives, which hurts performance badly. So we need to check the segment positioning on the lines as well.

2 Finding the closest points on two lines

2.1 Condition for the closest points

Consider two segments, P_1P_2 and Q_1Q_2. Let M and N be the closest points on the two straight lines that contain those segments, respectively, so that MN is the distance between the lines.[1]

We can write $\vec{M} = \vec{P}_1 + \mu\,\overrightarrow{P_1P_2}$ and $\vec{N} = \vec{Q}_1 + \nu\,\overrightarrow{Q_1Q_2}$ for some μ and ν. Since M and N are the *closest* points, they minimize the square of the distance, $\|\overrightarrow{MN}\|^2 = \overrightarrow{MN} \cdot \overrightarrow{MN} = \overrightarrow{MN}^2$, where

$$\begin{aligned}
\overrightarrow{MN} = \vec{N} - \vec{M} &= (\vec{Q}_1 + \nu\,\overrightarrow{Q_1Q_2}) - (\vec{P}_1 + \mu\,\overrightarrow{P_1P_2}) \\
&= \overrightarrow{P_1Q_1} + \nu\,\overrightarrow{Q_1Q_2} - \mu\,\overrightarrow{P_1P_2}
\end{aligned}$$

In the case where $\overrightarrow{MN}^2$ is minimized, μ and ν will satisfy

$$\begin{aligned}
0 &= \frac{\partial\,\overrightarrow{MN}^2}{\partial \mu} = 2\,\overrightarrow{MN} \cdot \frac{\partial\,\overrightarrow{MN}}{\partial \mu} & 0 &= \frac{\partial\,\overrightarrow{MN}^2}{\partial \nu} = 2\,\overrightarrow{MN} \cdot \frac{\partial\,\overrightarrow{MN}}{\partial \nu} \\
0 &= \left(\overrightarrow{P_1Q_1} - \mu\,\overrightarrow{P_1P_2} + \nu\,\overrightarrow{Q_1Q_2}\right) \cdot \overrightarrow{P_1P_2} & 0 &= \left(\overrightarrow{P_1Q_1} - \mu\,\overrightarrow{P_1P_2} + \nu\,\overrightarrow{Q_1Q_2}\right) \cdot \overrightarrow{Q_1Q_2} \\
0 &= \mu\,\overrightarrow{P_1P_2}^2 - \nu\,\overrightarrow{P_1P_2} \cdot \overrightarrow{Q_1Q_2} - \overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1} & 0 &= -\mu\,\overrightarrow{P_1P_2} \cdot \overrightarrow{Q_1Q_2} + \nu\,\overrightarrow{Q_1Q_2}^2 + \overrightarrow{Q_1Q_2} \cdot \overrightarrow{P_1Q_1} \qquad (2.1)
\end{aligned}$$

or, using matrix notation,

$$\begin{bmatrix} \overrightarrow{P_1P_2}^2 & -\overrightarrow{P_1P_2} \cdot \overrightarrow{Q_1Q_2} \\ -\overrightarrow{P_1P_2} \cdot \overrightarrow{Q_1Q_2} & \overrightarrow{Q_1Q_2}^2 \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} + \begin{bmatrix} -\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1} \\ \overrightarrow{Q_1Q_2} \cdot \overrightarrow{P_1Q_1} \end{bmatrix} = 0 \qquad (2.2)$$

[1] If the lines are coplanar, $M = N$, but that does not affect what follows. If the lines are identical, M and N aren't well defined, since the lines are "closest" at all points; this case may need to be treated specially.

2.2 Finding the closest points

Starting from equations (2.1),

$$0 = \mu\,\overrightarrow{P_1P_2}^2 - \nu\,\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2} - \overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1} \qquad 0 = -\mu\,\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2} + \nu\,\overrightarrow{Q_1Q_2}^2 + \overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}$$

we can eliminate ν:

$$\begin{aligned}
0 &= \left(\mu\,\overrightarrow{P_1P_2}^2 - \nu\,\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2} - \overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1}\right)\,\overrightarrow{Q_1Q_2}^2\\
0 &= \left(-\mu\,\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2} + \nu\,\overrightarrow{Q_1Q_2}^2 + \overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}\right)\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right)\\
0 &= \mu\left(\overrightarrow{P_1P_2}^2\,\overrightarrow{Q_1Q_2}^2 - \left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right)^2\right) - \left(\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1}\right)\,\overrightarrow{Q_1Q_2}^2 + \left(\overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}\right)\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right)
\end{aligned}$$

The expression multiplying μ can be simplified:

$$\overrightarrow{P_1P_2}^2\,\overrightarrow{Q_1Q_2}^2 - \left|\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right|^2 = l_P^2\,l_Q^2 - \left(l_P\,l_Q\ \cos\measuredangle(P,Q)\right)^2 = \left(l_P\,l_Q\ \sin\measuredangle(P,Q)\right)^2 = \left\|\overrightarrow{P_1P_2}\times\overrightarrow{Q_1Q_2}\right\|^2$$

resulting in

$$0 = \mu\left(\overrightarrow{P_1P_2}\times\overrightarrow{Q_1Q_2}\right)^2 - \left(\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1}\right)\,\overrightarrow{Q_1Q_2}^2 + \left(\overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}\right)\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right) \tag{2.3}$$

Likewise, we can eliminate μ:

$$\begin{aligned}
0 &= \left(\mu\,\overrightarrow{P_1P_2}^2 - \nu\,\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2} - \overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1}\right)\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right)\\
0 &= \left(-\mu\,\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2} + \nu\,\overrightarrow{Q_1Q_2}^2 + \overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}\right)\,\overrightarrow{P_1P_2}^2\\
0 &= \nu\left(\overrightarrow{P_1P_2}^2\,\overrightarrow{Q_1Q_2}^2 - \left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right)^2\right) + \left(\overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}\right)\,\overrightarrow{P_1P_2}^2 - \left(\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1}\right)\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right)\\
0 &= \nu\left(\overrightarrow{P_1P_2}\times\overrightarrow{Q_1Q_2}\right)^2 + \left(\overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}\right)\,\overrightarrow{P_1P_2}^2 - \left(\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1}\right)\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right) \qquad (2.4)
\end{aligned}$$

Interestingly, the terms multiplying μ and ν are the same.

3 Location checks for CCD without thickness

In the no-thickness case, a line/line collision will actually result in a segment/segment collision if M and N are located *within* the segments P_1P_2 and Q_1Q_2 respectively, i.e. if $\mu \in [0,1]$ and $\nu \in [0,1]$.

When handling CCD, we are working with a pair of *moving* segments, so we must consider $\vec{P}_i(t)$, $\vec{Q}_i(t)$ as linear expressions in time. That means we can't simply solve for μ and ν in general. But we can still verify if equations (2.3) and (2.4) have solutions for $\mu \in [0,1]$ and $\nu \in [0,1]$ for some "interesting" time interval $t \in [t_a, t_b]$. If we rewrite the equations as $w(t)\,\mu + m(t) = 0$, $w(t)\,\nu + n(t) = 0$, where

$$\begin{aligned}
w(t) &= \left(\overrightarrow{P_1P_2}\times\overrightarrow{Q_1Q_2}\right)^2\\
m(t) &= -\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1}\right)\,\overrightarrow{Q_1Q_2}^2 + \left(\overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}\right)\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right)\\
n(t) &= \left(\overrightarrow{Q_1Q_2}\cdot\overrightarrow{P_1Q_1}\right)\,\overrightarrow{P_1P_2}^2 - \left(\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1Q_1}\right)\left(\overrightarrow{P_1P_2}\cdot\overrightarrow{Q_1Q_2}\right)
\end{aligned}$$

then we can only have a successful collision if

$$w([t_a, t_b])\,\mu + m([t_a, t_b]) \ni 0 \qquad \text{for some } \mu \in [0,1] \qquad \Longrightarrow \qquad w([t_a, t_b]) * [0,1] + m([t_a, t_b]) \ni 0 \tag{3.1}$$
$$w([t_a, t_b])\,\nu + n([t_a, t_b]) \ni 0 \qquad \text{for some } \nu \in [0,1] \qquad \Longrightarrow \qquad w([t_a, t_b]) * [0,1] + n([t_a, t_b]) \ni 0 \tag{3.2}$$

where $f([a, b])$ denotes the image of the interval $[a, b]$ under f, and $*$ represents interval multiplication.

Since w, m and n are all quartic (4th degree) polynomials, finding the images of $[t_a, t_b]$ exactly requires solving cubic polynomials. At the time of this writing, we don't do that (due to the amount of implementation and testing work necessary), and use approximations to generate ain interval that merely contains the image instead. Switching to exactly solving would result in tighter bounds and thus better pruning.

4 Location checks for CCD with thickness

Not fully solved yet...

A Scratchpad =)

A.1 basics

To do: add no-thickness case. When you decompose $\overrightarrow{MN} = 0$ wrt $(\overrightarrow{P_1P_2},\ \overrightarrow{Q_1Q_2},\ \hat{n})$, you end up with the same result as the thickness case below.

For scalar case, solve. For intervals, not so much.

A.2 bad!

Do backwards: consider $\mu \in [0, 1]$, $\nu \in [0, 1]$. Do the equations allow zero in the result? Just need to really consider $\mu, \nu \in \{0, 1\}$ since it's all linear.

Do math as polynomials (4 cases for each of 2 eqns), get result intervals, combine per eqn.

$$\begin{aligned}
0 &= -\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1} & 0 &= \overrightarrow{Q_1Q_2} \cdot \overrightarrow{P_1Q_1} \\
0 &= \overrightarrow{P_1P_2}^2 - \overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1} & 0 &= -\overrightarrow{P_1P_2} \cdot \overrightarrow{Q_1Q_2} + \overrightarrow{Q_1Q_2} \cdot \overrightarrow{P_1Q_1} \\
0 &= -\overrightarrow{P_1P_2} \cdot \overrightarrow{Q_1Q_2} - \overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1} & 0 &= \overrightarrow{Q_1Q_2}^2 + \overrightarrow{Q_1Q_2} \cdot \overrightarrow{P_1Q_1} \\
0 &= \overrightarrow{P_1P_2}^2 - \overrightarrow{P_1P_2} \cdot \overrightarrow{Q_1Q_2} - \overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1} & 0 &= -\overrightarrow{P_1P_2} \cdot \overrightarrow{Q_1Q_2} + \overrightarrow{Q_1Q_2}^2 + \overrightarrow{Q_1Q_2} \cdot \overrightarrow{P_1Q_1}
\end{aligned} \tag{A.1}$$

Problem: doesn't work well. Decoupling results in way too many false positives.